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Best Approximation on Smooth Arcs

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Suppose that Γ is a Jordan arc in the complex plane and that f is continuous on Γ with modulus of continuity $\omega(f, t)$. Γ is said to have property J if for any such function there is a sequence of polynomials $\{P_n\}$, P_n of degree $< n$, such that

$$E_n(f) = \sup_{z \in \Gamma} |f(z) - P_n(z)| = O(\omega(f, 1/n)) \qquad \text{as} \quad n \to \infty.$$

That is, Γ has property J if the Jackson theorem, which holds for straight line segments, is true for Γ.

D. J. Newman has dealt with the problem of characterizing arcs which have property J. In [5] he conjectures that Γ has property J if and only if Γ is C^1. In this paper we show that this condition is not sufficient. We also obtain an analogous result for closed C^1 Jordan curves.

Let ψ be the conformal mapping of $|w| > 1$ onto the complement of Γ, such that for $a > 0$,

$$\psi(w) = aw + c_0 + (c_1/w) + (c_2/w^2) + \cdots.$$

The possession of property J is related to the smoothness of $\psi(w)$ for $|w| = 1$. Andersson [2] has shown that if Γ has property J, then $\psi(e^{i\theta}) \in \mathrm{Lip}(1 - \epsilon)$ for all $\epsilon > 0$. He uses this to show that the right-angled arc $[0, 1] \cup [0, i]$ does not have property J, since the mapping ψ is not $\mathrm{Lip}(\alpha)$ for $\alpha > \frac{1}{2}$. By refining Andersson's proof we shall prove the following:

THEOREM 1. *Suppose Γ has property J. Then $\psi(e^{i\theta})$ has modulus of continuity $\omega(t) = O(t \log^3(1/t))$.*

This will prove our assertion that $\Gamma \in C^1$ is not sufficient for property J, since for any n we can construct a curve for which $\omega(\psi, t) = k(t \log^n(1/t))$ for some positive constant k. Given n, there exists $a_n > 0$ such that the image of $[-a_n, a_n]$ under the mapping $g(z) = z \log^n(1/z)$ is a C^1 Jordan arc, which we call Γ. Here we take the branch cut for $\log z$ to be along the negative imaginary axis. Furthermore, if we let Ω be the upper half-disk of radius slightly larger than a_n, suitably smoothed at the corners, then g is a conformal mapping of Ω onto the interior D of a closed C^1 Jordan curve Γ_1, of which Γ is a subarc. The conformality follows from a lemma of Warschawski [6, p. 312], since Ω may be chosen so that $\operatorname{Re} g' > 0$ in the convex domain Ω. Now let h map the unit disk $|\zeta| < 1$ conformally onto Ω, so that $\phi = g \circ h$ is a conformal mapping of $|\zeta| < 1$ onto D, and ϕ is continuous for $|\zeta| \leqslant 1$. Since $\partial\Omega$ may be assumed to be at least C^2, h is Lipschitz continuous for $|\zeta| \leqslant 1$ so ϕ has modulus of continuity $= K'(t \log^n(1/t))$, for K' a constant.

Now let ψ be the conformal mapping of $|w| > 1$ onto the complement of Γ and let $D' = \psi^{-1}(D)$. Let H map $|\zeta| < 1$ conformally onto D'. Then $\phi_1 = \psi \circ H$ maps $|\zeta| < 1$ conformally onto D. But then $\phi_1 = \phi \circ T$ where T is a linear fractional transformation of $|\zeta| < 1$ onto itself. Finally we note that $\psi(e^{i\theta}) = g \circ h \circ T \circ H^{-1}(e^{i\theta})$ then has modulus of continuity equal to $K(t \log^n(1/t))$ in a neighborhood of $\psi^{-1}(0)$. Thus, $\psi(e^{i\theta})$ has this modulus of continuity.

In order to prove Theorem 1 we need refinements of two well-known theorems of Hardy and Littlewood. In the following proofs, $k_1, k_2, \ldots$ will denote positive constants.

LEMMA 1 [3, p. 125]. *Suppose $f(z) = u(z) + iv(z)$ is holomorphic in $|z| < 1$ and that $u(z)$ is continuous in $|z| \leqslant 1$. Suppose that $u(e^{it})$ has modulus of continuity $\omega(t)$. Then for $1 > r \geqslant \frac{1}{2}$ and all θ,*

$$|f'(re^{i\theta})| \leqslant 5 \int_{1-r}^{\pi} (\omega(t)/t^2)\, dt.$$

LEMMA 2 [3, p. 128]. *Suppose $f(z)$ is holomorphic in $|z| < 1$ and $f'(z)$ satisfies the condition*

$$|f'(re^{i\theta})| \leqslant M \int_{1-r}^{\pi} (\omega(t)/t^2)\, dt$$

for $1 > r \geqslant \frac{1}{2}$ and all θ. Here M is constant, $\omega(t)$ is nondecreasing, nonnegative, and bounded for $0 \leqslant t \leqslant \pi$, $t = O(\omega(t))$ and $\int_0^\pi (\omega(t)/t)\, dt < \infty$. Then $f(z)$ extends continuously to $|z| \leqslant 1$ and $f(e^{i\theta})$ has modulus of continuity

$$\omega^*(\theta) = O\left[\int_0^\theta \frac{\omega(t)}{t}\, dt + \theta \int_\theta^\pi \frac{\omega(t)}{t^2}\, dt\right].$$

Proof of Theorem 1. We let $f(z) = \bar{z}$, so that there exist k_1 and $\{P_n\}$ for which $|f(z) - P_n(z)| \leqslant k_1/n$ for all n and all $z \in \Gamma$.

Now for $|u| < 1$ we define

$$\tilde{f}(u) = \frac{1}{2\pi i}\int_{|w|=1} \frac{f\circ\psi(w)}{w-u}\,dw$$

and for $0 < x < 1$

$$\tilde{f}_x(u) = \tilde{f}(xu) = \frac{1}{2\pi i}\int_{|w|=1} \frac{f\circ\psi(w)}{w-xu}\,dw.$$

Next consider

$$P_{n,x}(u) = \frac{1}{2\pi i}\int_{|w|=1} \frac{P_n\circ\psi(w)}{w-xu}\,dw.$$

For fixed x, $P_{n,x}$ is a polynomial of degree less than n. Thus

$$\begin{aligned} E_n(\tilde{f}_x) &\leqslant \max_{|u|=1} \frac{1}{2\pi}\int_{|w|=1} \frac{|f\circ\psi(w) - P_n\circ\psi(w)|}{|w-xu|}\,|dw| \\ &\leqslant E_n(f)\max_{|u|=1}\frac{1}{2\pi}\int_{|w|=1}\frac{1}{|w-xu|}\,|dw| \\ &\leqslant \frac{k_2}{n}\log\frac{1}{1-x}. \end{aligned} \tag{1}$$

Thus, by the inverse theorem for approximation on the unit circle

$$\omega(\log^{-1}(1/(1-x))\,\tilde{f}_x(e^{i\theta}), \theta) = O(\theta\log(1/\theta)).$$

By Lemma 1, we then have

$$|\tilde{f}_x'(re^{i\theta})| \leqslant k_3\log(1/(1-x))\log^2(1/(1-r)).$$

Letting $x = r$,

$$\begin{aligned} |\tilde{f}'(r^2e^{i\theta})| &\leqslant k_3\log^3(1/(1-r)) \\ &\leqslant k_3\log^3((1+r)/(1-r^2)) \\ &\leqslant k_4\log^3(1/(1-r^2)). \end{aligned}$$

Replacing r^2 by r we conclude that

$$|\tilde{f}'(re^{i\theta})| \leqslant k_4\log^3(1/(1-r)).$$

Thus, $\tilde{f}$ satisfies the hypotheses of Lemma 2 with $\omega(t) = t\log^2 t$, so that $\tilde{f}$ extends continuously to the closed disk and $\tilde{f}(e^{i\theta})$ has modulus of continuity

$$\begin{aligned} \omega(\tilde{f}, \theta) &= O\left[\int_0^\theta \log^2 t\,dt + \theta\int_\theta^\pi (1/t)\log^2 t\,dt\right] \\ &= O(\theta\log^3(1/\theta)). \end{aligned}$$

But

$$f(u) = \frac{1}{2\pi i}\int_{|w|=1} \frac{\bar{\psi}(w)}{w-u}\,dw = \frac{1}{2\pi i}\int_{|w|=1} \frac{\bar{\psi}(1/w)}{w-u}\,dw = \bar{\psi}\left(\frac{1}{\bar{u}}\right) - \frac{a}{u}.$$

Thus the theorem is proved.

By a simple modification of the proof at step (1), one may easily prove the following

COROLLARY. *Suppose that for f continuous on Γ and k a positive integer*

$$E_n(f) = O(\log^k n\,\omega(f, 1/n)), \tag{2}$$

then $\psi(e^{i\theta})$ has modulus of continuity $\omega(t) = O(t \log^{k+3}(1/t))$.

Thus we see that the class of C^1 arcs is, in a sense, a long way from possessing property J, in that for each k there is a C^1 arc for which (2) cannot hold.

Suppose now that Γ is a closed Jordan curve. We shall say that Γ has property J_A if, for any function f which is analytic in D = Interior Γ and continuous on $\bar{D}$, there exists a sequence of polynomials $\{P_n\}$, P_n of degree $< n$, such that

$$E_n(f) = \sup_{z \in \bar{D}} |f(z) - P_n(z)| = O(\omega(f, 1/n)) \qquad \text{as} \quad n \to \infty.$$

Here $\omega(f, t)$ is the modulus of continuity of f on Γ. It follows from a result of Al'per [1] that any $C^{1+\delta}$ curve has property J_A. We have, however the following

THEOREM 2. *The class of closed C^1 Jordan curves does not have property J_A.*

Proof. Let Γ_0 be a C^1 Jordan arc for which the exterior mapping function ψ has modulus of continuity $= k_5(t \log^5(1/t))$, as constructed above. After a linear transformation, we may assume that -2 and $+2$ are endpoints of Γ_0, which lies in the z plane. We then write $z = w + 1/w$ and let Γ be the image of Γ_0 in w plane. Then Γ is a closed C^1 Jordan curve. As in [5], we note that for f continuous on Γ_0, approximation to $f(w + (1/w))$ on Γ by polynomials in w and $1/w$ yields ordinary polynomial approximation to $f(z)$ on Γ_0. Under the assumption that Γ has property J_A, we shall see that $f(z) = \bar{z}$ may be approximated sufficiently well on Γ_0 to apply the corollary. Thus we consider $g(w) = f(w + (1/w)) = \bar{w} + (1/\bar{w})$, which is Lipschitz continuous on Γ.

Now we define

$$\begin{aligned} \frac{1}{2\pi i}\int_\Gamma \frac{g(t)}{t-w}\,dt &= g_1(w) \qquad \text{for} \quad w \in \operatorname{Int} \Gamma, \\ &= g_2(w) \qquad \text{for} \quad w \in \operatorname{Ext} \Gamma. \end{aligned}$$

According to the Plemelj–Privalov formulas, g_1 and g_2 are continuous on $\overline{\operatorname{Int} \Gamma}$ and $\overline{\operatorname{Ext} \Gamma}$, respectively, and for $w \in \Gamma$

$$g_1(w) = \frac{1}{2} g(w) + \frac{1}{2\pi i} \int_\Gamma^{\text{P.V.}} \frac{g(t)}{t - w} \, dt,$$

$$g_2(w) = -\frac{1}{2} g(w) + \frac{1}{2\pi i} \int_\Gamma^{\text{P.V.}} \frac{g(t)}{t - w} \, dt.$$

Furthermore, on Γ, g_1 and g_2 both have modulus of continuity $= O(t \log(1/t))$. (See [4, p. 46]. This result is not stated in the theorem in [4], but it is in fact what is proved there.)

By hypothesis, g_1 may be approximated by polynomials on $\overline{\operatorname{Int} \Gamma}$ and, as a consequence of the hypothesis, g_2 is approximable on $\overline{\operatorname{Ext} \Gamma}$ by polynomials in $1/w$, with

$$E_n(g_k) = O((1/n) \log n), \qquad k = 1, 2.$$

But on Γ, $g(w) = g_1(w) - g_2(w)$, so g is approximable by polynomials in w and $1/w$ and is therefore approximable by polynomials in $w + (1/w)$. Therefore we have on Γ_0

$$E_n(f) = O((1/n) \log n).$$

It then follows from the argument of Theorem 1, as used in proving the Corollary, that

$$\omega(\psi(e^{i\theta})) = O(t \log^4(1/t)).$$

This being false, the theorem is proved.

Finally, we note that the class of closed C^1 Jordan curves is far from possessing property J_A in that for each k there is such a curve for which (2) cannot hold.

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